Modal Logic

Submitted to
Prof . Lubomír Popelínský,
Masaryk University

Prepared by master student
Abdullah Alshbatat

* Introduction
* Kripke's Formulation of Modal Logic
* Frames and Forcing
* Modal Tableaux
* Soundness and completeness
* Modal Axioms and special Accessibility Relations

Introduction

Modal Logic:

- Is the study of modal propositions and the logical relationships that they bear to one another. The most well-known are propositions about what is necessarily the case and what is possibly the case.
- Is an extension of classical propositional or predicate logic.
- Make precise the properties of possibility, necessity, belief, knowledge.
- Studies reasoning that involves the use of the expressions 'necessarily' and 'possibly'.

$\Box\varphi$ "it is necessary that φ ", " φ will always be true "

$\Diamond\varphi$ " it is possible that φ ", " φ will eventually be true "

Syntax:

Definition : A modal language $\mathcal{L}$ consists of the following disjoint sets of distinct primitive symbols:

1. **Variables**: $x, y, z, v, x_0, x_1, ..., y_0, y_1, ..., ...$ (an infinite set).
2. **Constants**: $c, d, c_0, d_0, ...$ (any set of them).
3. **Connectives**: $\wedge, \neg, \vee, \rightarrow, \leftrightarrow$.
4. **Quantifiers**: $\forall, \exists$.
5. **Predicate symbols**: $P, Q, R, P_1, P_2, ...$
6. **Function symbols**: $f, g, h, f_0, f_1, f_2, ..., g_2, ...$
7. **Basic operator** : $\Box, \Diamond$.
8. **Punctuation** : the comma, and the (right and left) parentheses) , (.

Definition : Formulas.

1. Every atomic formula is a formula.
2. If α, β are formulas, then so are $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, $(\neg \alpha)$, $(\alpha \vee \beta)$.
3. If ν is variable and α is formula, then $((\exists \nu)\ \alpha)$ and $((\forall \nu)\ \alpha)$ are also formulas.
4. If φ is a formula , then so are $(\Box \varphi)$ and $(\Diamond \varphi)$.

Definition :

1. **A Subformula** of a formula φ consecutive sequence of symbols from φ which itself formula.
2. **An occurrence** of a variable ν in a formula φ is **bound** if there is a subformula ϕ of φ containing that occurrence of ν such that ϕ begins with $((\exists \nu)(\forall \nu))$.

   An occurrence of ν in φ is free if it is not bound.
3. A variable ν is said to **occur free** in φ if it has at least one free occurrence there.
4. A **sentence** of Modal logic is a formula with no free occurrences of any variable.
5. An **open formula** is a formula without quantifiers.

Kripke's Formulation of Modal Logic

- Kripke have been introduced as means of giving semantics to modal logic, (introduced a domain of possible worlds).

- We consider W is collection of possible worlds. Each world $w \in W$ constitutes a view of reality as represent by structure C(w) associated with it.

-Modal Kripke introduced an accessibility relation on the possible worlds and this accessibility relation played a role in the definition of truth for modal sentences.

- We write w $\Vdash \varphi$ to mean φ is true in the possible world w. ("read as w forces φ" or "φ is true at w".)

If φ is a sentence of classical language, φ is true in the structure C(w).

If $\Box$ is interpreted as necessity, truth in all possible worlds.

If $\Diamond$ is interpreted as possibility, truth in some possible worlds.

Frames and Forcing

Semantics:

Definition: Let C = (W, S, $\{C(p)\}_{p \in W}$), consist of a set W, a binary relation S on W and function that assigns to each p in W a (classical) structure C(p) for $\mathcal{L}$.

We denote to the fact that the relation S holds between p and q as either pSq or $(p,q) \in S$.

We say C is frame for the language $\mathcal{L}$ ($\mathcal{L}$- *frame*) if for every p and q in W, pSq implies that $C(p) \subseteq C(q)$ and the interpretation of the constants in $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ are the same in C(p) as in C(q).

Definition (**Forcing for frames**): Let C = (W, S, {C(p) }$_{p \in W}$) be a frame for language $\mathcal{L}$, p be in W, and φ be a sentence of the language $\mathcal{L}(p)$. We give a definition of p forces φ, $p \Vdash \varphi$ by induction on sentence φ.

1. For atomic sentence φ, $p \Vdash \varphi \Leftrightarrow \varphi$ is true in C(p).
2. $p \Vdash (\varphi \rightarrow \psi) \Leftrightarrow p \Vdash \varphi$ implies $p \Vdash \psi$.
3. $p \Vdash (\neg \varphi) \Leftrightarrow p$ does not force φ (written) $p \nVdash \varphi$.
4. $p \Vdash ((\forall x)\, \varphi(x) \Leftrightarrow$ for every constant c in $\mathcal{L}(p)$, $p \Vdash \varphi(c)$.
5. $p \Vdash (\exists x)\, \varphi(x) \Leftrightarrow$ there is a constant c in $\mathcal{L}(p)$ such that $p \Vdash \varphi(c)$.
6. $p \Vdash (\varphi \wedge \psi) \Leftrightarrow p \Vdash \varphi$ and $p \Vdash \psi$.
7. $p \Vdash (\varphi \vee \psi) \Leftrightarrow p \Vdash \varphi$ or $p \Vdash \psi$. ($\Box \varphi$) and ($\Diamond \varphi$).
8. $p \Vdash \Box \varphi \Leftrightarrow$ for all $q \in$ W such that pSq, $q \Vdash \varphi$.
9. $p \Vdash \Diamond \varphi \Leftrightarrow$ there is a $q \in$ W such that pSq, $q \Vdash \varphi$.

Definition : Let φ be a sentence of the language $\mathcal{L}$. We say that φ is forced in the $\mathcal{L}$*- frame C*, $\Vdash_C \varphi$, if every p in W forces φ , We say φ is **valid** . $\models \varphi$, if φ is forced in every $\mathcal{L}$*- frame*.

Definition : Let Σ be a set of sentences in a modal language $\mathcal{L}$. and φ a single sentence of $\mathcal{L}$. φ is a **logical consequence** of Σ, $\Sigma \models \varphi$, if φ is forced in every $\mathcal{L}$ frame C in which every $\psi \in \Sigma$ is forced .

Modal Tableaux

For Modal Logic we begin with a signed forcing assertion $\mathrm{T}p \Vdash \varphi$ or $\mathrm{F}p \Vdash \varphi$, to build either frame agreeing with the assertion or decide that any such attempt leads to a contradiction.

- begin with $\mathrm{F}\,p \Vdash \varphi$; find either a frame in which p does not force φ or decide that we have a modal proof of φ.

Definition: Modal tableaux and tableau proofs:

are labeled binary trees. The labels (called entries of the tableau) are now either signed forcing assertions (i.e., labels of the form $\mathrm{T}p \Vdash \varphi$ or $\mathrm{F}p \Vdash \varphi$ for φ a sentence of any given appropriate language) or accessibility assertions $p\mathcal{S}q$.

We read $\mathrm{T}p \Vdash \varphi$ as p forces φ and $\mathrm{F}p \Vdash \varphi$ as p does not forces φ .

Definition: (Atomics tableaux): We begin by fixing a modal language $\mathcal{L}$ and an expansion to $\mathcal{L}_C$ given by adding new constant symbols $\mathbf{c}_i$ for $i \in \mathcal{N}$. In the tableaux, φ and ψ , if unquantified, are any sentences in the language $\mathcal{L}_C$. If quantified, they are formulas in which only **x** is free.

$T\ p \Vdash \varphi$ For any atomic sentence φ and any p	$F\ p \Vdash \varphi$ For any atomic sentence φ and any p
T $\vee$ $T\ p \Vdash \varphi \vee \psi$ branches: $T\ p \Vdash \varphi$ $T\ p \Vdash \psi$	F $\vee$ $F\ p \Vdash \varphi \vee \psi$ $F\ p \Vdash \varphi$ $F\ p \Vdash \psi$
F $\wedge$ $F\ p \Vdash \varphi \wedge \psi$ branches: $F\ p \Vdash \varphi$ $F\ p \Vdash \psi$	T $\vee$ $T\ p \Vdash \varphi \wedge \psi$ $T\ p \Vdash \varphi$ $T\ p \Vdash \psi$
T $\rightarrow$ $T\ p \Vdash \varphi \rightarrow \psi$ branches: $F\ p \Vdash \varphi$ $T\ p \Vdash \psi$	F $\rightarrow$ $F\ p \Vdash \varphi \rightarrow \psi$ $T\ p \Vdash \varphi$ $F\ p \Vdash \psi$

T ¬

$$T\ p \Vdash \neg\varphi$$
$$|$$
$$F\ p \Vdash \varphi$$

F ¬

$$F\ p \Vdash \neg\varphi$$
$$|$$
$$T\ p \Vdash \varphi$$

T ∃

$$T\ p \Vdash (\exists x)\varphi(x)$$
$$|$$
$$T\ p \Vdash \varphi(c)$$

For some new c

F ∃

$$F\ p \Vdash (\exists x)\varphi(x)$$
$$|$$
$$F\ p \Vdash \varphi(c)$$

For any appropriate c

T ∀

$$T\ p \Vdash (\forall x)\varphi(x)$$
$$|$$
$$T\ p \Vdash \varphi(c)$$

For any appropriate c

F ∀

$$F\ p \Vdash (\forall x)\varphi(x)$$
$$|$$
$$F\ p \Vdash \varphi(c)$$

For some new c

T □

$$T\ p \Vdash \Box\varphi$$
$$|$$
$$T\ q \Vdash \varphi$$

For any appropriate q

F □

$$F\ p \Vdash \Box\varphi$$
$$|$$
$$pSq$$
$$|$$
$$F\ q \Vdash \varphi$$

For some new q

T ◇

$$T\ p \Vdash \Diamond\varphi$$
$$|$$
$$pSq$$
$$|$$
$$T\ q \Vdash \varphi$$

For some new q

T ◇

$$T\ p \Vdash \Diamond\varphi$$
$$|$$
$$T\ q \Vdash \varphi$$

For any appropriate q

Definition: We fix a set $\{ p_i \mid i \in \mathcal{N} \}$ of potential candidates for the *p*'s and *q*'s in our forcing assertions.

A ***Modal tableau*** (for $\mathcal{L}$) is a binary tree labeled with signed forcing assertions or accessibility assertions; both sorts of labels are called entries of the tableau.

The class of modal tableaux (for $\mathcal{L}$) is defined inductively as follows.

1. Each atomic tableau T is a tableau.
   - in cases (T∃) and (F∀), *c* is new, means that *c* is on of the constants c_i added on to $\mathcal{L}$ to get $\mathcal{L}_C$ which does not appear in φ.
   - in (F∃) and (T∀), any appropriate *c* , means that any constant in $\mathcal{L}$ or φ.
   - in cases (F□) and (T◇), *q* is new; means that *q* is any of the p_i other than *p*.
   - in (T□) and (F◇), any appropriate *q*, means that the tableau is just $Tp \Vdash \Box \varphi$ or $Fp \Vdash \Diamond \varphi$ as there is no appropriate q.

2. If $\mathcal{T}$ is a finite tableau, P a path on $\mathcal{T}$, E an entry of $\mathcal{T}$ occurring on P and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by adjoining an atomic tableau with root entry E to $\mathcal{T}$ at the end of the path P, then $\mathcal{T}'$ is also a tableau.

- c in (T∃) and (F∀), is on of the constants c_i that do not appear in any entry on $\mathcal{T}$.
- appropriate c in (F∃) and (T∀), any c in $\mathcal{L}$ or appearing in an entry on P of the form Tq ⊩ ψ or Fq ⊩ ψ such that $q S p$ also appears on P.
- in (F□) and (T◇), q is new; means that we choose a p_i not appearing in $\mathcal{T}$ as q.
- in (T□) and (F◇), appropriate q; means we can choose any q such that $p S q$ is an entry on P.

3. If $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n, \ldots$ is a sequence of finite tableaux such that, for every $n \geq 0$, $\mathcal{T}_{n+1}$ is constructed from $\mathcal{T}_n$ by an application of 2, Then $\mathcal{T} = \bigcup \mathcal{T}_n$ is also a tableau.

Definition (Tableau Proofs)**:** Let $\mathcal{T}$ be a modal tableau and P a path in $\mathcal{T}$.

1) P is **contradictory** if , for some forcing assertion $p \Vdash \varphi$, both T $p \Vdash \varphi$ and F $p \Vdash \varphi$ appear as entries on P.

2) $\mathcal{T}$ is **contradictory** if every path through $\mathcal{T}$ is contradictory.

3) $\mathcal{T}$ is a **proof** of φ if $\mathcal{T}$ is finite contradictory modal tableau with its root node labeled F p $\Vdash \varphi$ for some p. φ is provable, $\vdash \varphi$ if there is a proof of φ.

* If there is any contradictory tableau with root node F $p \Vdash \varphi$, then there is one that is finite, i.e., a proof of φ: just terminate each path when it becomes contradictory.

* When construct proofs, Mark any contradictory path with the symbol $\otimes$ and terminate the development of the tableau along that path.

Example 1: $\varphi \rightarrow \Box \varphi$

1	$F\ w \Vdash \varphi \rightarrow \Box \varphi$		
2	$T\ w \Vdash \varphi$		by 1
3	$F\ w \Vdash \Box \varphi$		by 1
4	wSv	for a new v	by 3
5	$F\ v \Vdash \varphi$		by 3

This failed attempt at a proof suggests a frame counterexamples C for which W={w,v}, S={(w,v)} , φ is true at w but not at v. $\varphi \rightarrow \Box \varphi$ is not valid.

Example 2: $\Box \varphi \rightarrow \varphi$

1	$F\ w \Vdash \Box \varphi \rightarrow \varphi$	
2	$T\ w \Vdash \Box \varphi$	by 1
3	$F\ w \Vdash \varphi$	by 1

The frame counterexamples consists of a one world W={w} with empty accessibility relation S and φ false at w.
$\Box \varphi \rightarrow \varphi$ is not valid.

Various interpretations of $\Box$ might tempt one to think that $\Box \varphi \rightarrow \varphi$ should be valid, Why?

Example 3: $\Box (\forall x) \varphi(x) \rightarrow (\forall x) \Box \varphi(x)$

1	$F\ w \Vdash \Box (\forall x) \varphi(x) \rightarrow (\forall x) \Box \varphi(x)$	
2	$T\ w \Vdash \Box (\forall x) \varphi(x)$	by 1
3	$F\ w \Vdash (\forall x) \Box \varphi(x)$	by 1
4	$F\ w \Vdash \Box \varphi(c)$	by 3
5	wSv	by 4
6	$F\ v \Vdash \varphi(c)$	by 4
7	$T\ v \Vdash (\forall x) \varphi(x)$	by 2, 5
8	$T\ v \Vdash \varphi(c)$	by 7
	$\otimes$	by 6, 8

Example 4:

$(\forall x)\neg\Box\varphi \rightarrow \neg\Box(\exists x)\varphi$

- The frame counterexample consists of world W={w,v}, S={(w,v)} , constant domain C = {c, d}; and no atomic sentence true at w and φ(d) true at v.

- $(\forall x)\neg\Box\varphi \rightarrow \neg\Box(\exists x)\varphi$ is not valid.

1	$F\ w \Vdash (\forall x)\neg\Box\varphi \rightarrow \neg\Box(\exists x)\varphi$		
2	$T\ w \Vdash (\forall x)\neg\Box\varphi$		by 1
3	$F\ w \Vdash \neg\Box(\exists x)\varphi$		by 1
4	$T\ w \Vdash \Box(\exists x)\varphi$		by 3
5	$T\ w \Vdash \neg\Box\varphi(c)$		by 2
6	$F\ w \Vdash \Box\varphi(c)$		by 5
7	wSv		by 6
8	$F\ v \Vdash \varphi(c)$		by 6
9	$T\ v \Vdash (\exists x)\varphi$		by 4, 7
10	$T\ v \Vdash \varphi(d)$	new d	by 9

Definition (Modal tableaux from Σ): a set of sentence of a modal language called premises, the same modal tableaux except that we allow one additional formation rule:

- If τ is finite tableau from Σ, $\varphi \in \Sigma$, P a path in τ and p a possible world appearing in some signed forcing assertion on P, then appending T $p \Vdash \varphi$.

We write $\Sigma \vdash \varphi$ to denote that φ is provable from Σ.

Example : tableau proof of $\Box \forall x \varphi(x)$ from the premise $\forall x \varphi(x)$.

1	$F\, p \Vdash \Box (\forall x) \varphi(x)$	
2	pSq	by 1
3	$F\, q \Vdash (\forall x) \varphi(x)$	by 1
4	$F\, q \Vdash \varphi(c)$	new c by 3
5	$T\, q \Vdash (\forall x) \varphi(x)$	premise
6	$T\, q \Vdash \varphi(c)$	by 5
	$\otimes$	

Soundness and completeness

* Our goal here is to show that in modal logic provability implies validity.

* In modal logic we must define a set W of possible world and, for each $p \in$ W, a structure based on constants occurring on the path.

* W will consist of the *p's* occurring in signed forcing assertions along the path.

* The accessibility relation on W will then be defined by the assertions pSq occurring on the path.

Definition: suppose C = (V, T, C(p)) is a frame for a modal language $\mathcal{L}$, τ is a tableau whose root is labeled with a forcing assertion about a sentence φ of $\mathcal{L}$ and P is a path through τ.

W set of *p's* appearing in forcing assertions on P and S the accessibility relation on Wdetermined by the assertions pSq occurring on P.

We say that C **agrees with** P if there are maps f and **g** such that:

1. f is a map from W into V that preserve the accessibility relation, i.e., $p\mathrm{S}q \Rightarrow f(p)\,\mathrm{T}\,f(q)$.

2. $\boldsymbol{g}$ sends each constant c occurring in any sentence ψ of a forcing assertion T $p \Vdash \psi$ or F $p \Vdash \psi$ on P to a constant in $\mathcal{L}(f(p))$. $\boldsymbol{g}$ is the identity on constants of $\mathcal{L}$.

also extend **g** to be a map on formulas in the obvious way: To get **g**(ψ) replace every constant c in ψ by **g**(c).

3. If T $p \Vdash \psi$ is on P, then $f(p)$ forces **g**(ψ) in C and if F $p \Vdash \psi$ is on P then f(p) does not force **g**(ψ) in C.

Theorem : suppose C = (V, T, C(p)) is a frame for a modal language $\mathcal{L}$, and τ is a tableau whose root is labeled with a forcing assertion about a sentence φ of $\mathcal{L}$. if $q \in$ V and either

1. F r⊩ φ is the root of et of τ and q does not force φ in C .

Or

2. T r⊩ φ is the root of et of τ and q does force φ in C .

Then there is a path P through τ that agrees with C with a witness function f that sends r to q.

Theorem : (**Soundness,** $\vdash \varphi \Rightarrow \models \varphi$) If there is a (modal) tableau proof of a sentence φ (of a modal logic), then φ is (modally) valid.

Theorem : (**Completeness**, $\models \varphi \Rightarrow \vdash \varphi$) If a sentence φ of modal logic is valid (in the frame semantics), then it has a (modal)tableau proof.

Theorem (**Soundness**, $\Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi$) If there is a (modal) tableau proof of φ from a set Σ of sentences, then φ is logical consequence of Σ.

Theorem (**Completeness**, $\Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi$) If φ is logical consequence of a set Σ of sentences of modal logic, then there is a modal tableau proof of φ from Σ.

Modal Axioms and special Accessibility Relations

- Some particular intended interpretation of modal operator might suggest axioms that one might wish to add to modal logic.

Example: if $\Box$ means "it is necessarily true that" or "I know that" one might want to include an axiom scheme asserting $\Box\varphi \rightarrow \varphi$ for every sentence φ .

but if $\Box$ intended to mean "I believe that", then we might well reject $\Box\varphi \rightarrow \varphi$ as an axiom: I can have false beliefs.

- There are close connections between certain natural restriction on the accessibility relation in frames and various common axioms for modal logic.
- It is possible to formulate precise equivalents (the sentences forced in all frames with specified type of accessibility relation are precisely the logical consequences of some axiom system).

Definition :

1. Let $\mathcal{F}$ be a class of frames and φ a sentence of modal language $\mathcal{L}$. We say that φ is $\mathcal{F}$- ***valid***, $\models_{\mathcal{F}} \varphi$, if φ is forced in every frame C ∈ $\mathcal{F}$.

2. Let F be a rule or a family of rules for developing tableaux, The **F**- tableaux extended to include the formation rules in F. As well as F-tableau is proof of sentence φ if it is finite, has a root node of the form F$p \Vdash \varphi$ and every path is contradictory. We say that φ is F-provable, $\vdash_{F} \varphi$, if it has an F-tableau proof.

Definition:

1. $\mathcal{R}$ is the class of all **reflexive frames**, i.e., all frames in which the accessibility relation is reflexive (wSw holds for every w ∈ W).

2. R is the **reflexive tableau development rule** that says that, given a tableau τ, we may form a new tableau τ' by adding wSw to the end of any path P in τ on which w occurs.

3. $\mathcal{T}$ is the set of universal closures of all instances of the scheme T: $\Box\varphi \rightarrow \varphi$.

Theorem : For any sentence φ of our modal language $\mathcal{L}$, the following conditions are equivalent:

1. $\mathcal{T} \models \varphi$, φ is a logical consequence of $\mathcal{T}$.
2. $\mathcal{T} \vdash \varphi$, φ is a tableau provable from $\mathcal{T}$.
3. $\models_{\mathcal{R}} \varphi$, φ is forced in every reflexive $\mathcal{L}$-*frame*.
4. $\vdash_{\mathcal{R}} \varphi$, φ is provable with the reflexive tableau development rule.

Lemma :

1. if T $p \Vdash \Box\psi$ appear on P and $pS'q$, Then T $q \Vdash \psi$ appears on P.
2. if F $p \Vdash \Diamond\psi$ appear on P and $pS'q$, Then F q $\Vdash \psi$ appears on P.

Example : (Introspection and Transitivity): the scheme PI, $\Box\varphi \to \Box\Box\varphi$. It is called the scheme of positive introspection as it expresses the view that *what I believe, I believe I believe*.

There is no contradictory. By reading off the true atomic statement from the tableaux, we get a three-world frame C= (W, S, C(p)). With W={w, v , u}, S= { (v, u),(w,v) }, C(v) $\models \varphi$ and C(u), C(w) $\not\models \varphi$.

1	F $w \Vdash \Box\varphi \to \Box\Box\varphi$		
2	T $w \Vdash \Box\varphi$		by 1
3	F $w \Vdash \Box\Box\varphi$		by 1
4	wSv	new v	by 3
5	F $v \Vdash \Box\varphi$		by 3
6	vSu	new u	by 5
7	F $u \Vdash \varphi$		by 5
8	T $v \Vdash \varphi$		by 2, 4

Definition :

1. $\mathcal{TR}$ is the class of all **transitive frames**, i.e., all frames C=(W,S,C(p)) in which S is transitive: wSv ∧ vSu ⇒ wSu.

2. TR is the **transitive tableau development rule** that says that if wSv and vSu appear on a path P of tableau τ, then we can produce another tableau τ' by appending wSu to the end of P.

Theorem: For any sentence φ of our modal language $\mathcal{L}$, the following conditions are equivalent:

1. $\mathcal{PI} \models \varphi$, φ is a logical consequence of $\mathcal{PI}$.
2. $\mathcal{PI} \vdash \varphi$, φ is a tableau provable from $\mathcal{PI}$.
3. $\models_{\mathcal{TR}} \varphi$, φ is forced in every transitive *$\mathcal{L}$-frame*.
4. $\vdash_{\mathrm{TR}} \varphi$, φ is provable with the transitive tableau development rule.

Definition:

1. $\mathcal{E}$ is the class of all **Euclidean frames**, i.e., all frames C=(W,S,C(*p*)) in which S is **Euclidean** : wSv ∧ wSu ⇒ uSv.

2. E is the **Euclidean tableau development rule** which says that if wSv and wSu appear on a path P of tableau $\mathcal{T}$, then we can produce another tableau $\mathcal{T}'$ by appending uSv to the end of P.

3. $\mathcal{NI}$ is the set of all universal closures of instances of the scheme NI: $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$.

Theorem : For any sentence φ of our modal language $\mathcal{L}$, the following conditions are equivalent:

1. $\mathcal{NI} \models \varphi$, φ is a logical consequence of $\mathcal{NI}$.
2. $\mathcal{NI} \vdash \varphi$, φ is a tableau provable from $\mathcal{NI}$.
3. $\models_{\mathcal{E}} \varphi$, φ is forced in every Euclidean $\mathcal{L}$-*frame*.
4. $\vdash_{E} \varphi$, φ is provable with the Euclidean tableau development rule.

Definition:

1.$\mathcal{SE}$ is the class of all **serial frames**, i.e., all frames C=(W,S,C(p)) in which there is, for every $p \in$ W, a q such that pSq.

2. SE is the **serial tableau development rule** which says that if p appear on a path P of tableau $\mathcal{T}$, then we can produce another tableau $\mathcal{T}'$ by appending pSq to the end of P for a new q.

3. $\mathcal{D}$ is the set of all universal closures of instances of the scheme D: $\Box\varphi \rightarrow \neg\Box\varphi$.

Theorem: For any sentence φ of our modal language $\mathcal{L}$, the following conditions are equivalent:

1. $\mathcal{D} \models \varphi$, φ is a logical consequence of $\mathcal{D}$.
2. $\mathcal{D} \vdash \varphi$, φ is a tableau provable from $\mathcal{D}$.
3. $\models_{\mathcal{SE}} \varphi$, φ is forced in every serial $\mathcal{L}$-*frame*.
4. $\vdash_{\text{SE}} \varphi$, φ is provable with the serial tableau development rule.

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